Int. J. Solids Structures Vol. 29, No. 14/15, pp. 1939–1945, 1992
Printed in Great Britain.
0020-7683/92 $5.00 + .00
Pergamon Press plc

AN ANALYTIC SOLUTION FOR INTERACTING INTERFACE CRACKS IN ANISOTROPIC DISSIMILAR MATERIALS

L. Ni† and S. Nemat-Nasser
Center of Excellence for Advanced Materials, Department of Applied Mechanics and Engineering Sciences, University of California, San Diego, La Jolla, CA 92093, U.S.A.

Abstract—An analytic study of possible interface failure by coalescence of an interface microcrack and a primary interface crack in anisotropic dissimilar materials is presented. The stress intensity factor of mode I at the tip of an interface macrocrack is shown to be always greater than that at the tip of a neighboring interface microcrack, suggesting that the primary crack is expected to grow into the microcrack.

1. INTRODUCTION

In a recent paper of the authors (Ni and Nemat-Nasser, in press), a complete solution for a single fully-open interface crack in anisotropic dissimilar materials was presented. The same approach can be applied to find the solution for multiple interface cracks in anisotropic dissimilar materials. As an example, in this paper, we consider two interacting interface cracks, one a macrocrack and the other a microcrack, in anisotropic dissimilar materials with uniform tractions applied far from the cracks. Analytic solutions for the dislocation distribution and interfacial tractions are obtained. For a special case of the mode I deformation, possible interface failure by coalescence of an interface microcrack and a primary interface macrocrack is examined in detail. It is shown that the stress intensity factor of mode I at the tip of an interface macrocrack is always greater than that at the tip of a neighboring interface microcrack, suggesting that the primary crack is expected to grow into the microcrack.

2. FORMULATION

Consider two interface cracks, one a primary macrocrack and the other a microcrack, lying at the interface along the $x_2 = 0$-plane between two dissimilar anisotropic elastic solids (Fig. 1). The elastic tensors are C^+_{ijkl} and C^-_{ijkl}, for $x_2 > 0$ and $x_2 < 0$, respectively. On the x_2-plane, let the primary interface crack extend from $x_1 = a_2$ to $x_1 = b_2$, and the interface microcrack extend from $x_1 = a_1$ to $x_1 = b_1$, where $-\infty < a_1 < b_1 < a_2 < b_2 < \infty$. The crack edges are all parallel to the x_3-axis. All field variables are assumed to be functions of x_1 and x_2 only. Uniform normal tractions, T, in-plane shear tractions, S, and anti-plane shear tractions, J, are applied far from the interface cracks, in the x_2-, x_1-, and x_3-directions, respectively.

As discussed in Ni and Nemat-Nasser (1991a; in press), equilibrium requires

$$\nabla \cdot \boldsymbol{\sigma} = \nabla \cdot [\mathbf{C}^{\pm} : \nabla \mathbf{u}] = \mathbf{0}, \tag{1a}$$

where $\mathbf{u}$ and $\boldsymbol{\sigma}$ denote the displacement and the stress fields, and ∇ is the gradient operator. The interface tractions and displacements must satisfy

† On leave from Academia Sinica, Beijing, China.

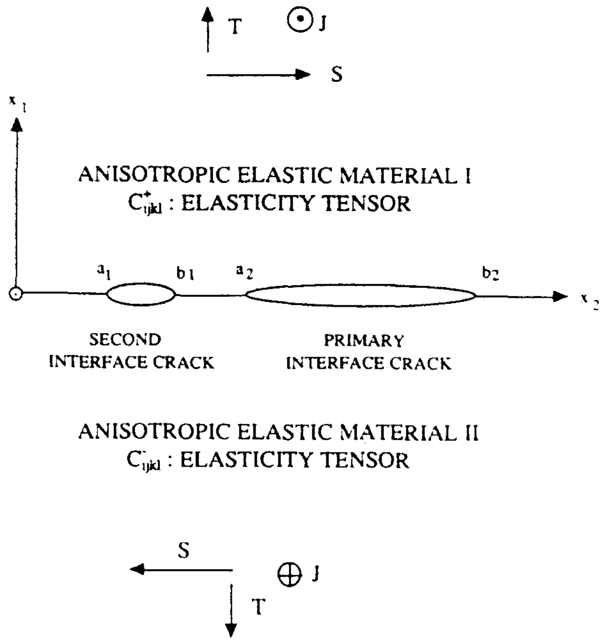

Fig. 1.

$$\mathbf{C}^+ : \nabla\mathbf{u}(x_1, 0^+) = \mathbf{C}^- : \nabla\mathbf{u}(x_1, 0^-) = \mathbf{t}(x_1), \tag{1b}$$

$$\mathbf{u}(x_1, 0^+) - \mathbf{u}(x_1, 0^-) = -\int_{a}^{x_1} \mathbf{B}(\xi)\,\mathrm{d}\xi, \tag{1c}$$

$$\mathbf{u}(x_1, x_2) \to \mathbf{0} \quad |x_2| \to \infty, \tag{1d}$$

where $\mathbf{B}(x_1)$ is the dislocation density vector, and $\mathbf{t}(x_1)$ is the interfacial traction vector. Then, Fourier transform yields the basic relation between $\mathbf{t}(x_1)$ and $\mathbf{B}(x_1)$ (Willis, 1971):

$$\mathbf{t}(x_1) = \frac{1}{\pi}\mathrm{Re}\left[\frac{\mathbf{\Lambda}}{x_1 - 0i}\right] * \mathbf{B}(x_1), \tag{2a}$$

where $*$ denotes the convolution integral; the positive-definite matrix $\mathbf{\Lambda}$ may be expressed in terms of Stroh's matrices (Stroh, 1958):

$$\mathbf{\Lambda} = -i[\bar{\mathbf{A}}_+\bar{\mathbf{L}}_+^{-1} - \mathbf{A}_-\mathbf{L}_-^{-1}]^{-1}, \tag{2b}$$

with

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \qquad \mathbf{L} = [\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3]; \tag{2c, d}$$

and

$$\begin{bmatrix} \mathbf{a}_i \\ \mathbf{l}_i \end{bmatrix}, \qquad i = 1, 2, 3,$$

are the six-dimensional eigenvectors of the matrix

$$\mathbf{N} = -\begin{bmatrix} \mathbf{T}^{-1}\mathbf{R}^T & \mathbf{T}^{-1} \\ \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T-\mathbf{Q} & \mathbf{R}\mathbf{T}^{-1} \end{bmatrix}, \tag{2e}$$

where

$$\mathbf{Q} = [c_{j1k1}], \qquad \mathbf{R} = [c_{j1k2}], \qquad \mathbf{T} = [c_{j2k2}]; \tag{2f–h}$$

the subscripts + and − correspond to the upper and lower half-spaces.

When the cracks are fully open, the total tractions over the crack surfaces L are zero, where L denotes the union of $L_1 = [a_1, b_1]$ and $L_2 = [a_2, b_2]$. Then, we obtain

$$\mathbf{t}(x_1)+\boldsymbol{\tau} = \mathbf{0} \tag{3}$$

for x_1 in L and $\boldsymbol{\tau} = (S, T, J)$. Combining (2a) and (3), it follows that

$$\frac{1}{\pi}\,\mathrm{Re}\left[\frac{\mathbf{A}}{x_1-0i}\right] * \mathbf{B}(x_1)+\boldsymbol{\tau} = \mathbf{0} \tag{4}$$

for x_1 in L. Equivalently, (4) can be rewritten as a system of multiple integral equations:

$$\frac{1}{\pi}\mathbf{A}_1\left\{\int_{L_1}\frac{\mathbf{B}(\xi)}{x_1-\xi}\,\mathrm{d}\xi+\int_{L_2}\frac{\mathbf{B}(\xi)}{x_1-\xi}\,\mathrm{d}\xi\right\}+\mathbf{A}_2\mathbf{B}(x_1)+\boldsymbol{\tau} = \mathbf{0}, \tag{5}$$

for x_1 in $L = L_1+L_2$, and $\mathbf{A}_1 = \mathrm{Re}\,\mathbf{A}$, $\mathbf{A}_2 = -\mathrm{Im}\,\mathbf{A}$.

The consistency conditions require that

$$\int_{L_1}\mathbf{B}(\xi)\,\mathrm{d}\xi = \mathbf{0}, \qquad \int_{L_2}\mathbf{B}(\xi)\,\mathrm{d}\xi = \mathbf{0}, \tag{6a, b}$$

and

$$\mathbf{B}(x_1) = \mathbf{0}, \tag{6c}$$

when x_1 is not in L, which state the fact that, outside the crack zones, the gap, the tangent shift along the x_1-axis, and the anti-plane shift along the x_3-axis are all zero.

3. SOLUTION

One way to solve (5) for the dislocation density vector $\mathbf{B}(x_1)$, under (6a–c), is as follows. First, we use the method developed by Ni and Nemat-Nasser (1991a; in press) to decouple the system (4) of Cauchy singular integral equations with constant coefficients, then we directly apply to each of the decoupled equations the standard method for solving Cauchy integral equations with a contour consisting of multiple simple arcs; see, e.g., Muskhelishvili (1953), Mikhlin (1964) and Erdogan (1978). After simple manipulation, the dislocation distribution and the interfacial tractions are obtained,

$$\mathbf{B}(x_1) = \mathrm{sgn}\,(x_1-a_2)\,\frac{\mathbf{E}}{\sqrt{1-\beta^2}}\begin{bmatrix} G(x_1,m) & 0 & 0 \\ 0 & \bar{G}(x_1,m) & 0 \\ 0 & 0 & G(x_1,\tfrac{1}{2}) \end{bmatrix}\mathbf{E}^{-1}\mathbf{A}_1^{-1}\boldsymbol{\tau},$$

$$\text{for } x_1 \text{ in } L = L_1+L_2; \tag{7}$$

$$\mathbf{t}(x_1) = \operatorname{sgn}(x_1-a_1)(x_1-a_2)]\mathbf{\Lambda}_1\mathbf{E}\begin{bmatrix} G(x_1,m) & 0 & 0 \\ 0 & \bar{G}(x_1,m) & 0 \\ 0 & 0 & G(x_1,\frac{1}{2}) \end{bmatrix}\mathbf{E}^{-1}\mathbf{\Lambda}_1^{-1}\boldsymbol{\tau},$$

$$\text{for } x_1 \text{ not in } L = L_1+L_2. \quad (8)$$

In (7) and (8),

$$\beta = \left(1-\frac{|\mathbf{\Lambda}|}{|\mathbf{\Lambda}_1|}\right)^{1/2}, \quad (9a)$$

or equivalently

$$\beta = [-\tfrac{1}{2}\operatorname{tr}(\mathbf{W}\mathbf{D}^{-1})^2]^{1/2}, \quad (9b)$$

is the generalized Dundurs constant (Ting, 1986; Ni and Nemat-Nasser, in press); matrices **D** and **W** are defined by Ting (1986);

$$\mathbf{E} = [\nu_1, \nu_2, \nu_3], \quad (9c)$$

with ν_j as the three-dimensional eigenvectors of $-i\mathbf{\Lambda}_1^{-1}\mathbf{\Lambda}_2$ corresponding to the eigenvalues β_j, $j = 1, 2, 3$, ($\beta_1 = \beta$, $\beta_2 = -\beta$, $\beta_3 = 0$). If we set

$$\mathbf{\Lambda} = \begin{bmatrix} \alpha_1 & \alpha_3+i\alpha_4 & \alpha_5+i\alpha_6 \\ \alpha_3-i\alpha_4 & \alpha_2 & \alpha_7+i\alpha_8 \\ \alpha_5-i\alpha_6 & \alpha_7-i\alpha_8 & \alpha_9 \end{bmatrix}, \quad (9d)$$

then ν_j are evaluated to be

$$\nu_1 = \begin{bmatrix} \lambda^2(\alpha_5\alpha_7-\alpha_3\alpha_9)-i\lambda(\alpha_6\alpha_7-\alpha_5\alpha_8-\alpha_4\alpha_9)+\alpha_6\alpha_8 \\ \lambda^2(\alpha_1\alpha_9-\alpha_5^2)-\alpha_6^2 \\ \lambda^2(\alpha_3\alpha_5-\alpha_1\alpha_7)+i\lambda(\alpha_3\alpha_6-\alpha_4\alpha_5-\alpha_1\alpha_8)+\alpha_4\alpha_6 \end{bmatrix} = \lambda|\Lambda|\begin{bmatrix} \lambda D_{12}-iW_{12} \\ \lambda D_{22} \\ \lambda D_{32}+iW_{23} \end{bmatrix} + \alpha(1-\lambda^2)\nu_3, \quad (9e)$$

$$\nu_2 = \bar{\nu}_1, \quad \nu_3 = \begin{bmatrix} \alpha_8 \\ -\alpha_6 \\ \alpha_4 \end{bmatrix} = -|\Lambda|D\begin{bmatrix} W_{23} \\ -W_{13} \\ W_{12} \end{bmatrix}, \quad (9f, g)$$

where D_{ij} and W_{ij} are elements of matrices **D** and **W**. The function $G(x_1, m)$ in (7) and (8) can be expressed explicitly,

$$G(x_1,m) = \frac{[(x_1-a_1)(x_1-a_2)+m(a_1+a_2-b_1-b_2)(x_1-a_2)+g]}{\sqrt{|x_1-a_1|\,|x_1-a_2|\,|x_1-b_1|\,|x_1-b_2|}}\left\{\frac{|(x_1-a_1)(x_1-a_2)|}{|(x_1-b_1)(x_1-b_2)|}\right\}^{\varepsilon_0}, \quad (9h)$$

where

$$m = \tfrac{1}{2} + i\gamma_0, \qquad \gamma_0 = \frac{1}{2\pi}\ln\left\{\frac{|1+\beta|}{|1-\beta|}\right\}. \tag{9i,j}$$

and the constant g is found to be

$$g = (a_2-b_2)\left(\frac{a_2-b_1}{b_2-b_1}\right)^m \left\{F_1\left[1+m, -m, m, 2; \frac{(a_2-b_2)}{(a_2-a_1)}, \frac{(a_2-b_2)}{(a_2-b_1)}\right]\right.$$

$$\left. + m(a_1+a_2-b_1-b_2)F_1[1+m, 1-m, m, 2; \frac{(a_2-b_2)}{(a_2-a_1)}, \frac{(a_2-b_2)}{(a_2-b_1)}\right]\bigg\}$$

$$\times \left\{F\left[m, 1-m; 1; \frac{(b_2-a_2)(b_1-a_1)}{(a_2-a_1)(b_2-b_1)}\right]\right\}^{-1}, \tag{9k}$$

with F the hypergeometric function, and F_1 the hypergeometric function of two variables.

4. CONCLUSION

If the in-plane and the anti-plane deformations are not coupled, then the solutions of the dislocation distribution vector $\mathbf{B}(x_1)$ and the interfacial traction vector $\mathbf{t}(x_1)$ for the case of the only in-plane deformation are simplified as

$$\mathbf{B}(x_1) = \frac{\operatorname{sgn}(x_1-a_2)/\alpha_0\alpha_{00}}{\sqrt{|x_1-a_1|\,|x_1-a_2|\,|x_1-b_1|\,|x_1-b_2|}} \times \left\{S\begin{bmatrix}\alpha_2(A\cos t - B\sin t)\\ \alpha_0(A\sin t + B\cos t) - \alpha_3(A\cos t - B\sin t)\end{bmatrix} - T\begin{bmatrix}\alpha_0(A\sin t + B\cos t) + \alpha_3(A\cos t - B\sin t)\\ -\alpha_1(A\cos t - B\sin t)\end{bmatrix}\right\},$$

for x_1 in L; (10a)

$$\mathbf{t}(x_1) = \frac{\operatorname{sgn}[(x_1-a_1)(x_1-a_2)]/\alpha_0}{\sqrt{|x_1-a_1|\,|x_1-a_2|\,|x_1-b_1|\,|x_1-b_2|}} \times \left\{S\begin{bmatrix}\alpha_0(A\cos t - B\sin t) + \alpha_3(A\sin t + B\cos t)\\ \alpha_2(A\sin t + B\cos t)\end{bmatrix} + T\begin{bmatrix}-\alpha_1(A\sin t + B\cos t)\\ \alpha_0(A\cos t - B\sin t) - \alpha_3(A\sin t + B\cos t)\end{bmatrix}\right\},$$

for x_1 not in L, (10b)

where

$$\alpha_0 = (\alpha_1\alpha_2 - \alpha_3^2)^{1/2}, \qquad \alpha_{00} = (\alpha_1\alpha_2 - \alpha_3^2 - \alpha_4^2)^{1/2}, \tag{10c,d}$$

$$A = A(x_1) = (x_1-a_1)(x_1-a_2) + \tfrac{1}{2}(a_1+a_2-b_1-b_2) + \mathrm{Re}\,[g], \tag{10e}$$

$$B = B(x_1) = \gamma_0(a_1+a_2-b_1-b_2) + \mathrm{Im}\,[g], \tag{10f}$$

and

$$t = \gamma_0 \ln \left\{ \frac{|(x_1-a_1)(x_1-a_2)|}{|(x_1-b_1)(x_1-b_2)|} \right\}. \tag{10g}$$

We further simplify the problem by assuming that: (i) there is only mode I deformation, i.e. $S = 0$, $T \neq 0$; (ii) $\beta = 0$. Then $\gamma_0 = 0$, and $m = 1/2$, which imply that there is no oscillation involved in the solutions of the dislocation density and the interfacial traction. Under these assumptions, from (10a, b) we obtain the dislocation density and the interfacial traction vectors, as follows:

$$\mathbf{B}(x_1) = \frac{\operatorname{sgn}(x_1-a_2)TA(x_1)\begin{bmatrix} -\alpha_3 \\ \alpha_1 \end{bmatrix}}{\alpha_0\sqrt{|x_1-a_1|\,|x_1-a_2|\,|x_1-b_1|\,|x_1-b_2|}}, \tag{11a}$$

for x_1 in L, and

$$\mathbf{t}(x_1) = \frac{\operatorname{sgn}[(x_1-a_1)(x_1-a_2)]TA(x_1)\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\sqrt{|x_1-a_1|\,|x_1-a_2|\,|x_1-b_1|\,|x_1-b_2|}}, \tag{11b}$$

for x_1 not in L, where

$$A(x_1) = (x_1-a_1)(x_1-a_2) + \tfrac{1}{2}(a_1+a_2-b_1-b_2)(x_1-a_2) + g, \tag{11c}$$

$$g = \tfrac{1}{2}(a_2-a_1)(a_2-b_1) - \tfrac{1}{2}(a_2-a_1)(b_2-b_1)E(\lambda)/K(\lambda), \tag{11d}$$

with

$$\lambda = \left[\frac{(b_2-a_2)(b_1-a_1)}{(a_2-a_1)(b_2-b_1)}\right]^{1/2}; \tag{11e}$$

$K(\lambda)$ and $E(\lambda)$ are the complete elliptic integrals of the first and second kind.

We now set $|b_1-a_1| = \varepsilon$, $|a_2-b_1| = k\varepsilon$, $|b_2-a_2| = 1$, and $\varepsilon < 1$, $k > 0$ (Fig. 2), which means that the primary crack is a macrocrack, and the other interface crack is a microcrack; the distance between the microcrack and the macrocrack, depending on k and ε, can be arbitrary. It now follows that

$$K_I^{(a_2)} = \sqrt{\frac{\pi(1+k)}{2k}}[(1+k\varepsilon)E(\lambda)/K(\lambda)-k\varepsilon]T, \tag{12a}$$

$$K_I^{(b_1)} = \sqrt{\frac{\pi(1+k\varepsilon)}{2k}}[(1+K)E(\lambda)/K(\lambda)-k]T. \tag{12b}$$

The ratio between the stress intensity factor of mode I at the tip $x_1 = a_2$ of the macrocrack and that at the tip $x_1 = b_1$ of the microcrack is

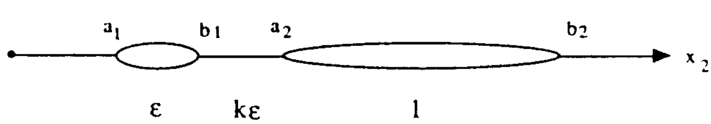

Fig. 2.

$$\frac{K_I^{a_2}}{K_I^{b_1}} = \frac{\sqrt{1+k}\,[(1+k\varepsilon)E(\lambda)/K(\lambda)-k\varepsilon]}{\sqrt{1+k\varepsilon}\,[(1+k)E(\lambda)/K(\lambda)]},$$
$$= \frac{\sqrt{1+k}\,[E(\lambda)/K(\lambda)-k\varepsilon(1-E(\lambda)/K(\lambda))]}{\sqrt{1+k\varepsilon}\,[E(\lambda)/K(\lambda)-k(1-E(\lambda)/K(\lambda))]}, \tag{13}$$

which is always greater than 1 for any $k > 0$ and $\varepsilon < 1$, since $E(\lambda)/K(\lambda) < 1$, where $\lambda = [(1+k)(1+k\varepsilon)]^{-1/2}$. Therefore,

$$K_I^{a_2} > K_I^{b_1} \tag{14}$$

always holds.

We conclude that the stress intensity factor of mode I at the tip of an interface macrocrack is always greater than that at the tip of a neighboring microcrack; hence the primary crack tends to grow into the microcrack.

Acknowledgement—This work has been supported by the U.S. Army Research Office under Contract No. DAAL-03-86-K-0169, to the University of California, San Diego.

REFERENCES

Erdogan, F. (1978). Mixed boundary-value problems in mechanics. In *Mechanics Today*, Vol. 4 (Edited by S. Nemat-Nasser), pp. 1–86. Pergamon Press, Oxford.
Mikhlin, S. G. (1964). *Integral Equations*. Pergamon Press, Oxford.
Muskhelishvili, N. I. (1953). *Singular Integral Equations*. Noordhoff, Groningen.
Ni, L. and Nemat-Nasser, S. (1991a) Interface crack in anisotropic dissimilar materials: an analytic solution. *J. Mech. Phys. Solids* **39**, 113–144.
Ni, L. and Nemat-Nasser, S. (In press). Interface cracks in anisotropic dissimilar materials: general case. *Q. Appl. Math.*
Stroh, A. N. (1958). Dislocations and cracks in anisotropic elasticity. *Phil. Mag.* **3**, 625–646.
Ting, T. C. T. (1986). Explicit solution and invariance of singularities at an interface crack in anisotropic composites. *Int. J. Solids Structures* **22**, 965–983.
Willis, J. R. (1971). Fracture mechanics of interfacial cracks. *J. Mech. Phys. Solids* **19**, 353–368.